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BAKALÁŘSKÁ PRÁCE

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Rozklady propojovacích sítí na dlouhé cesty

Katedra teoretické informatiky a matematické logiky

Vedoucí bakalářské práce:	Mgr. Petr Gregor, Ph.D.
Studijní program:	Informatika
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Název práce: Rozklady propojovacích sítí na dlouhé cesty

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Katedra: Katedra teoretické informatiky a matematické logiky

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Abstrakt: V této práci jsou shrnuty vlastnosti vybraných propojovacích sítí, jimiž jsou modifikace hyperkrychle. Konkrétně studujeme toleranci k chybám v tzv. augmentovaných krychlích s ohledem na existenci hamiltonovských kružnic. Také předvedeme metodu používající jednoduché prostředky lineární algebry, pomocí které lze zkonstruovat velké množství podgrafů n-dimenzionální augmentované krychle AQ_n izomorfních n-dimenzionální hyperkrychli Q_n. Dokážeme, že AQ_n s f vadnými hranami obsahuje kopii Q_n s nejvýše $\frac{n}{2n-1}f$ vadnými hranami. Pomocí tohoto výsledku jsme schopni některé vlastnosti Q_n s vadnými hranami přenést na AQ_n s (více) vadnými hranami. Podobným způsobem také dokážeme, že jestliže $f \leq 3n-7$ a každý vrchol AQ_n sousedí aspoň se dvěma zdravými hranami, pak AQ_n obsahuje hamiltonovskou kružnici tvořenou pouze zdravými hranami. Navíc dokážeme, že každé dva monomorfismy G_1 do AQ_n a G_2 do AQ_m lze složit na monomorfismus kartézského součinu $G_1 \,\square\, G_2$ do AQ_{n+m}.

Klíčová slova: hyperkrychle, augmented cube, hamiltonovské kružnice, vadné hrany

Title: Partitions of interconnection networks into long paths

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Abstract: We survey some basic properties of selected interconnection networks, more precisely modifications of hypercubes. In particular, we study fault-tolerance of augmented cubes with respect to the existence of Hamiltonian cycles. We also present a method to construct many spanning subgraphs of AQ_n isomorphic to Q_n using elementary facts from linear algebra. We prove that n-dimensional augmented cube AQ_n with f faulty edges contains a copy of n-dimensional hypercube Q_n with at most $\frac{n}{2n-1}f$ faulty edges. Using this result we are able to easily transfer several properties of Q_n with faulty edges to AQ_n with (more) faulty edges. Applying similar method, we also show that if $f \leq 3n-7$ and every vertex of AQ_n is incident to at least two non-faulty edges, then AQ_n has a non-faulty hamiltonian cycle. Moreover, we show that every two monomorphisms of G_1 to AQ_n and G_2 to AQ_m can be composed into a monomorphism of the Cartesian product $G_1 \,\square\, G_2$ to AQ_{n+m}.

Keywords: hypercube, augmented cube, hamiltonicity, faulty edges

Contents

Introduction

Many of today's applications such as weather prediction or applications of artificial intelligence require large amount of processing power. To meet these requirements much more powerful computers are needed. One way to achieve this is to create faster electronic components. There are however limiting factors for processor speed (i. e. the switching times of transistors, the temperature, the speed of light). So it appears that the most promising way forward is to use parallelism. If several operations can be performed simultaneously then the total computation time is reduced. One way of communication between processors is to use a shared memory and shared variables. However this is unrealistic for large numbers of processors. A more realistic assumption is that each processor has its own private memory and data communication takes place using message passing via an interconnection network.

It is well known that the topological structure of interconnection networks can be modeled by a graph with vertices representing processors and edges representing links between them. It is of great interest to study properties of graph models of interconnection networks. There are many requirements in designing the interconnection network (such as vertex degree, regularity, robustness, communication cost, network expansion cost). Some of these requirements are contradictory (such as communication cost, network expansion cost). Thus, it is not possible to build an interconnection network that would be optimal for all purposes. One has to design a suitable network depending on its properties and requirements.

One of important issues in designing interconnection networks is the possibility of routing. Routing is the process of selecting paths in a network along which to send network traffic. Routing schemes differ in their delivery semantics. One-to-one (unicast) routing delivers a message from one vertex to a single specific vertex. One-to-many (broadcast, resp. multicast) routing delivers a message from one vertex to all (resp. multiple) vertices. Many-to-many routing delivers messages from a group of vertices to another group of vertices. There are routing algorithms in various networks that find the shortest path (or paths) between the vertices (resp. vertex and set of vertices, resp. two sets of vertices). Sometimes on the contrary the largest possible number of visited vertices is required for multicast. In this case long paths and cycles can be used. They are also suitable for simple algorithms. The path and cycle embedding properties of many interconnection networks have been widely investigated.

We survey some basic properties of graph models of selected interconnection networks, more precisely modifications of hypercubes. In particular, we study fault-tolerance of augmented cubes with respect to the existence of Hamiltonian cycles. One of the main results of this thesis is Theorem 4.11 which says that n-dimensional augmented cube AQ_n with f faulty edges contains a copy of n-dimensional hypercube Q_n with at most $\frac{n}{2n-1}f$ faulty edges. This results was obtained using elementary facts from linear algebra. Using this theorem we are able to easily transfer some properties of faulty Q_n with $F \subseteq E(Q_n)$ to AQ_n with more faulty edges. For example we prove that AQ_n with $f \leq 2n-3$ contains a non-faulty hamiltonian cycle.

We also prove that if $f \leq 3n-7$ and every vertex of AQ_n is incident to at least two non-faulty edges, then AQ_n has a subgraph G with at most $2n-5$ faulty edges and every vertex of G is incident to at least two non-faulty edges. This implies that if $f \leq 3n-7$ and every vertex of AQ_n is incident to at least two non-faulty edges, then AQ_n contains a non-faulty hamiltonian cycle.

Moreover, we propose a method to obtain a monomorphism $h : V(G_1 \Box G_2) \rightarrow V(AQ_{m+n})$ as a composition of monomorphisms $g_1 : V(G_1) \rightarrow AQ_n$ and $g_2 : V(G_2) \rightarrow AQ_m$. This method may be used to find more subgraphs of augmented cubes isomorphic to hypercube and many other subgraphs. All these results are new, as far as we know, and they extend or improve previously known results, see Section 5.1.

Hypercube and its modifications are surveyed in Chapter 2, also some new propositions are mentioned in this Chapter. Paths and cycles in faulty hypercubes are surveyed in more detail in Chapter 3. Chapter 4 consist of new results on hypercubes in augmented cubes. Both previous and new results on paths and cycles in faulty augmented cubes are stated in Chapter 5. In Chapter 6 we propose the way to compose two monomorphisms to augmented cube.

1. Preliminaries

For standard terminology in graph theory we refer to [1]. A graph G of order n is said to be *k-pancyclic* (for $k \leq n$) if it contains a cycle of every length from k to n, and *pancyclic* if it is g-pancyclic, where $g = g(G)$ is the girth of G. A graph G of order n is *vertex-pancyclic* (resp. *edge-pancyclic*) if every vertex (resp. edge) lies on a k-cycle for every k from $g(G)$ to n. Obviously, an edge-pancyclic graph is vertex-pancyclic. Since a bipartite graph contains no odd cycles, the concept of bipancyclicity is proposed instead of pancyclicity. A bipartite graph G of order n is *bipancyclic* (also called *even-pancyclic*) if it contains a cycle of every even length from $g(G)$ to n.

A graph G is *hamiltonian connected* if there exists a hamiltonian path between any two vertices of G. A bipartite graph is *hamiltonian laceable* if there is a hamiltonian path between any two vertices from different partite sets.

Some of the links or processors in the interconnection network may be busy or broken. These links (edges) and processors (vertices) are called *faulty*. A graph G is said to be *faulty* if it has at least one faulty vertex or edge. The set of faulty vertices and edges is denoted by F. We study under what conditions some property of G is preserved in $G - F$. The fault tolerance is one of the major factors in evaluating the performance of networks.

A graph G is *k-fault-tolerant hamiltonian* (resp. connected, hamiltonian connected, pancyclic) if G is hamiltonian (resp. connected, hamiltonian connected, pancyclic) and $G - F$ remains hamiltonian (resp. connected, hamiltonian connected, pancyclic) for every $F \subseteq V(G) \cup E(G)$ with $|F| \leq k$. A graph G is *k-vertex-fault-tolerant hamiltonian* (resp. connected, hamiltonian connected, pancyclic) if G is hamiltonian (resp. connected, hamiltonian connected, pancyclic) and $G - F$ remains hamiltonian (resp. connected, hamiltonian connected, pancyclic) for every $F \subseteq V(G)$ with $|F| \leq k$. A graph G is *k-edge-fault-tolerant hamiltonian* (resp. connected, hamiltonian connected, pancyclic) if G is hamiltonian (resp. connected, hamiltonian connected, pancyclic) and $G - F$ remains hamiltonian (resp. connected, hamiltonian connected, pancyclic) for every $F \subseteq E(G)$ with $|F| \leq k$.

Some of the studied graphs can be constructed by the Cartesian product of two graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The *Cartesian product* of G_1 and G_2, denoted by $G_1 \,\square\, G_2$, is the graph with the vertex set $V_1 \times V_2$ such that (u_1, v_1) is connected to (u_2, v_2) if and only if either

- $u_1u_2 \in E(G_1)$ and $v_1 = v_2$, or
- $u_1 = u_2$ and $v_1v_2 \in E(G_2)$.

A *homomorphism* h of a graph G_1 to a graph G_2 is a mapping $h : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ implies $h(u)h(v) \in E(G_2)$. An injective homomorphism is called a *monomorphism*.

2. Hypercube modifications

The hypercube is one of the most frequently studied interconnection networks for parallel computers. It has many suitable properties (such as low diameter, high connectivity, regular degree etc.) and is so versatile that it can efficiently emulate a wide variety of other frequently used networks. Some important properties of the hypercube relevant to parallel computing can be found in [29]. However, properties of the hypercube are not suitable for all applications which inspired researchers to propose various hypercube modifications.

In this thesis almost all interconnection networks (except the balanced hypercube) are formed by a graph with the vertex-set $V = \{0,1\}^n$. We usually represent vertices by binary strings. If we need to index the bits of the vertex, they are always indexed from right to left; that is, we write $x = x_n x_{n-1} \dots x_1$ where $x_i \in \{0,1\}$.

Some of these hypercube modifications are defined recursively. The following notation will be useful. Let G_n be a graph with the vertex-set $V(G_n) = \{0,1\}^n$ and the edge-set E and let $u = u_n u_{n-1} \dots u_1$ be a vertex of G_n. Then $0u = 0u_n u_{n-1} \dots u_1$ and $0G_n$ denotes the graph with the vertex-set $V(0G_n) = \{0v; v \in V(G_n)\}$ and two vertices $0u, 0v \in V(0G_n)$ are connected if and only if $uv \in E(G_n)$. Analogously we define the graph $1G_n$.

In what follows we survey hypercube modifications and their basic properties. Some of the basic properties are stated without a proof, namely when it can be easily deduced.

2.1 Hypercube

Definition. *The n-dimensional hypercube Q_n is the graph with the vertex-set $V = \{0,1\}^n$ and two vertices are linked with an edge if and only if they differ in exactly one coordinate.*

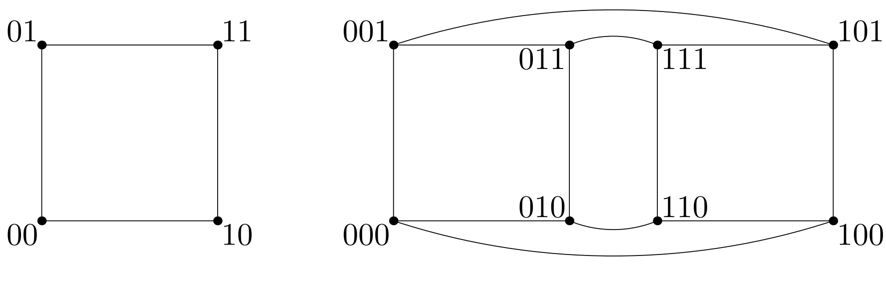

Figure 2.1: The hypercubes Q_2 and Q_3.

The edges in *dimension i* are the edges between vertices that differ in the i-th coordinate.

The hypercube Q_n is an n-regular n-connected bipartite graph with 2^n vertices and the diameter n. It is both vertex-transitive and edge-transitive. A well known result says that it has $2^n n!$ automorphisms formed uniquely by a composition of a permutation of dimensions and a translation.

Let $s = s_n s_{n-1} \dots s_1$ be a string of length n with $s_i \in \{0, 1, *\}$ for every i and let G be a graph with $V(G) = \{0,1\}^n$. Then $G(s)$ denotes the subgraph of the graph G induced by vertices matching the string s. The vertex $v = v_n v_{n-1} \dots v_1$ *matches* the string s if $v_i = s_i$ or $s_i = *$ for each i satisfying $1 \leq i \leq n$. Clearly, the subgraph $Q_n(s)$ is isomorphic to Q_k where k is the number of $*$'s in s and it is called a k-dimensional *subcube* of Q_n. On the other way, every subgraph of Q_n isomorphic to Q_k is $Q_n(s)$ for some string $s \in \{0, 1, *\}^n$.

Some other properties of hypercubes are discussed in Chapter 3.

2.2 Folded hypercube

Definition. *The n-dimensional folded hypercube FQ_n for $n \geq 2$ is the graph obtained from Q_n by adding all so called complementary edges connecting vertices $x = x_n x_{n-1} \cdots x_1$ and $\bar{x} = \bar{x}_n \bar{x}_{n-1} \cdots \bar{x}_1$ for each $x \in V(Q_n)$, where $\bar{x}_i = 1 - x_i$. Furthermore, $FQ_1 = Q_1$.*

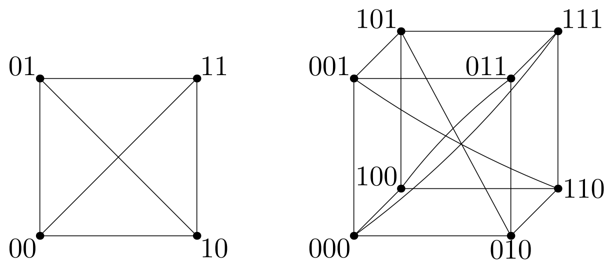

Figure 2.2: The folded hypercubes FQ_2 and FQ_3.

The folded hypercube was first proposed by El-Amawy and Latifi [13]. It has 2^n vertices. It is $(n + 1)$-regular and $(n + 1)$-connected. It has a diameter of $\lceil \frac{n}{2} \rceil$.

It is vertex-transitive. The number of automorphisms of FQ_n is $(n + 1)!2^n$ as was shown by Morteza Mirafzal [33]. Xu and Ma [42] showed that it is bipartite if and only if n is odd. Wang [38] showed $FQ_n - F$ with $F \subset E(FQ_n)$ and $|F| \leq n - 1$ is a hamiltonian graph for all $n \geq 2$.

Complementary edges join the vertices that differ in all bits of their label, so $FQ_n(s)$ is isomorphic to Q_k for every string $s \in \{0, 1, *\}^n$ with $k < n$ stars in s.

2.3 Augmented cube

The augmented cube AQ_n was proposed by Choudum and Sunitha [7]. It is defined as follows.

Definition. *The augmented cube AQ_n has the vertex-set $V = \{0, 1\}^n$ and two vertices u and v are joined with an edge if they differ in exactly one coordinate or exactly in a suffix; that is,*

$$E(AQ_n) = E(Q_n) \cup \{\{ws, w\bar{s}\}; w \in \{0, 1\}^{n-i}, s \in \{0, 1\}^i, 1 < i \leq n\}.$$

The edges between vertices that differ exactly in a suffix are called *suffix edges* and the edges between vertices that differ in exactly one bit, except those that differ in the last bit, are called *hypercube edges*.

The augmented cube of dimension n is a $(2n - 1)$-regular graph with 2^n vertices. The graph AQ_n is $(2n - 1)$-connected for every $n \neq 3$. Xiang and Stewart

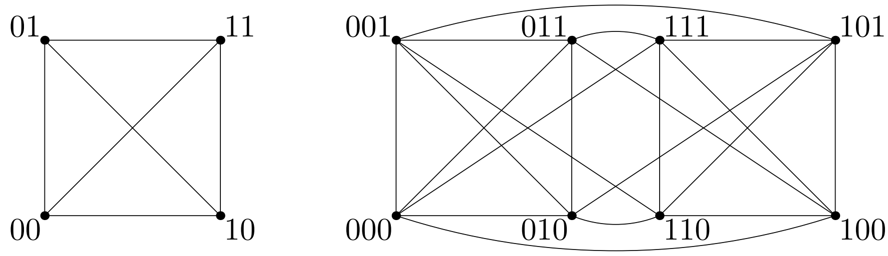

Figure 2.3: The augmented cubes AQ_2 and AQ_3.

[40] showed that it is vertex transitive. Choudum and Sunitha [8] showed that the number of automorphisms of AQ_n is 2^{n+3} for every $n \geq 4$.

The subgraph of AQ_n induced by the vertices with a fixed prefix of length l is isomorphic to AQ_{n-l} for every $0 \leq l < n$. For every $s \in \{0, 1, *\}^n$ with k stars, the subgraph $AQ_n(s)$ contains Q_k as a subgraph but it may contain more edges according to the following proposition.

Proposition 2.1. *Let $s \in \{0, 1, *\}^n$ with $1 \leq k < n$ stars in s, let p be the position of the rightmost non-star bit of s. Then $AQ_n(s)$ is isomorphic to the Cartesian product of Q_{k-p+1} and AQ_{p-1}.*

Proof. For $v \in V(AQ_n(s))$ the graph $AQ_n(s)$ does not contain any vertex that differs from v in coordinate p. Thus, $AQ_n(s)$ does not contain suffix edges uv with $v = u_n \dots u_{j+1} \bar{u}_j \bar{u}_{j-1} \dots \bar{u}_1$ and $j \geq p$. However, it contains all the hypercube edges in dimensions $d \in \{i; s_i = *\}$. Then the vertex $v \in V(AQ_n(s))$ is connected to the vertices with

1. the same suffix of length $p - 1$ that differ in exactly one bit on position $p_2 > p$,

2. the same prefix of length $k - p + 1$ that differ exactly in a suffix of length $1 < l < p$ or exactly in one bit on position $p_1 < p$.

Note that $V(AQ_n(s)) = \{0, 1\}^{p-1} \times \{0, 1\}^{k-p+1}$ and the described set of vertices correspond to the definition of Cartesian product of Q_{k-p+1} and AQ_{p-1}. $\square$

Corrollary 2.2. *Let $s \in \{0, 1, *\}^n$ and let $1 \leq k < n$ be the number of stars in s and let p be the position of the rightmost non-star bit of s. Then the degree of each vertex of $AQ_n(s)$ is $2n - k - p - 2$ for $p \geq 3$ and it is $n - k$ for $p \in \{1, 2\}$.*

Some new results on augmented cubes are described in Chapters 4 and 5.

2.4 Crossed cube

The crossed cube of dimension n, denoted by CQ_n, was first proposed by Efe [11]. It is defined as follows. Two binary strings $x = x_2x_1$ and $y = y_2y_1$ are *pair-related* (denoted by $x \sim y$) if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$.

Definition. *The crossed cube CQ_n of dimension n is a graph with the vertex-set $V = \{0, 1\}^n$. Two vertices $x = x_n \cdots x_2x_1$ and $y = y_n \cdots y_2y_1$ are connected with an edge if and only if there is j, $1 \leq j \leq n$, satisfying the following conditions:*

1. $x_n \cdots x_{j+1} = y_n \cdots y_{j+1}$,
2. $x_j \neq y_j$,
3. $x_{j-1} = y_{j-1}$, *if* j *is even, and*
4. $x_{2i}x_{2i-1} \sim y_{2i}y_{2i-1}$ *for each* $i = 1, 2, \ldots, \lceil \frac{j}{2} \rceil - 1$.

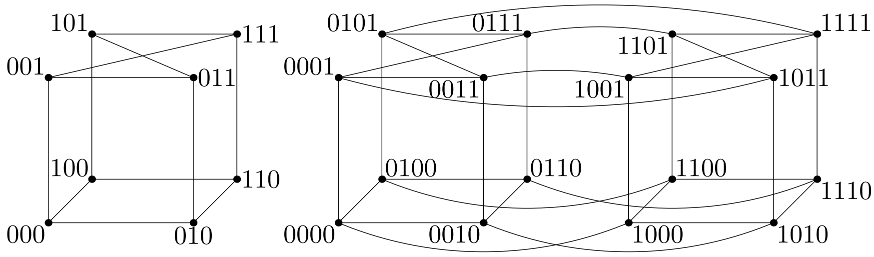

Figure 2.4: The crossed cubes CQ_3 and CQ_4.

The crossed cube of dimension n is an n-regular, n-connected graph with 2^n vertices. Its diameter is $\lceil \frac{n+1}{2} \rceil$. CQ_n is not vertex transitive for any $n \geq 5$, as was proved by Kulasinghe and Bettayeb [28]. It is not edge transitive for any $n \geq 3$. It is edge-pancyclic for every $n \geq 2$ [15].

CQ_n contains many subgraphs isomorphic to a crossed cube of lower dimension. As a result, CQ_n may be partitioned into isomorphic subgraphs either by removing the leftmost bit from the vertices or by removing any evenly indexed bit or any pair of adjacent bits $l_{2k+1}l_{2k}$ or any combination of these, as was shown by Efe et al. [12].

2.5 Twisted cube

Hilbers et al. [22] defined twisted cube TQ_n only for odd n. For a vertex $x = x_n x_{n-1} \cdots x_1$ we define a parity function $P_i : \{0,1\}^n \to \{0,1\}$ where $1 \leq i \leq n$, by $P_i(x) = x_i \oplus x_{i-1} \oplus \cdots \oplus x_1$, where $\oplus$ is the addition over $\mathbb{F}_2$. Let x^i be the vertex $x^i = x_n x_{n-1} \ldots \bar{x}_i \ldots x_1$.

TQ_n is built by replacing some edges from Q_n with edges that span two dimensions.

Definition. *The twisted cube* TQ_n *where* n *is odd has the vertex-set* $V = \{0,1\}^n$. *If* $P_{2j-1}(x) = 0$ *for some* $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, *we replace the edge between vertices* x *and* x^{2j} *with the edge between vertices* x *and* $(x^{2j})^{2j+1}$.

TQ_n is an n-regular n-connected graph with 2^n vertices and diameter $\lceil \frac{n+1}{2} \rceil$.

TQ_n is pancyclic for any odd integer $n \geq 3$, as was proved by Chang et al. [3]. Xu and Ma showed that TQ_n is also vertex-pancyclic for every odd $n \geq 3$ [43]. This result was improved by Fan et al. [16] who showed that TQ_n is edge-pancyclic for every odd $n \geq 3$.

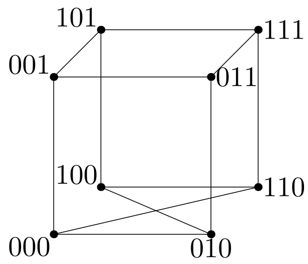

Figure 2.5: The twisted cube TQ_3.

2.6 Locally twisted cube

The n-dimensional locally twisted cube, denoted by LTQ_n, was defined by Yang et al. [47] for every $n \geq 2$ recursively as follows.

Definition. *1. $LTQ_2 = Q_2$,*

2. LTQ_n for $n \geq 3$ is built from $0LTQ_{n-1}$ and $1LTQ_{n-1}$ by connecting every vertex $x = 0x_{n-1}x_{n-2}\cdots x_1$ from $0LTQ_{n-1}$ with the vertex $1(x_{n-1} \oplus x_1)x_{n-2}\cdots x_1$ from $1LTQ_{n-1}$ by an edge, where $\oplus$ represents the addition over $\mathbb{F}_2$.

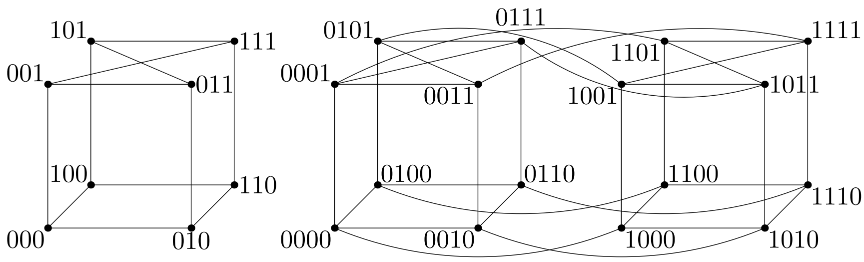

Figure 2.6: The locally twisted cubes LTQ_3 and LTQ_4.

The graph LTQ_n is an n-connected n-regular graph with diameter $D \leq \lceil \frac{n+3}{2} \rceil$ for every $n \geq 2$ [47]. LTQ_n is edge-pancyclic for every $n \geq 3$, as was proved by Yang et al. [48]. For every two vertices x and y at distance d in LTQ_n ($n \geq 3$) there is a path between x and y of each length from d to $2^n - 1$ except $d+1$. The subgraph of LTQ_n induced by all vertices with the same prefix of length k is isomorphic to LTQ_{n-k}. If $s = s_n *^{n-1}$ with $s_n \in \{0,1\}$, then the graph $LTQ_n(s)$ is isomorphic to Q_{n-1}.

Proposition 2.3. *Let $s \in \{0,1,*\}^n$ contain $n-1$ stars and $s_i \in \{0,1\}$, $1 < i < n$, then $LTQ_n(s)$ with $n > 3$ is isomorphic to a subgraph of LTQ_{n-1}, it contains 2^{n-1} vertices and $2^{n-3}(2n-3)$ edges. The degree of the vertices v with $v_1 = 0$ is $n-1$, the degree of the other vertices is $n-2$.*

Proof. According to the definition, the vertex $v \in V(LTQ_n)$ is connected to the following vertices:

- v^1, v^2 (since $LTQ_2 = Q_2$) for every v,
- v^i, for every $2 < i \leq n$ if $v_1 = 0$,

- $(v^i)^{i-1}$, for every $2 < i \leq n$ if $v_1 = 1$.

First we prove that $LTQ_n(s)$ is isomorphic to a subgraph of LTQ_{n-1}. Let $V = \{v \in \{0,1\}^{n-1}; v_j = u_j \text{ for } j < i, v_j = u_{j+1} \text{ for } i \leq j \leq n-1, u \in V(LTQ_n)\}$. It is clear that $V = V(LTQ_{n-1})$. $LTQ_n(s)$ is a subgraph of LTQ_n so it may only contain edges from $E(LTQ_n)$. If $i > 2$ then $E(LTQ_n(s))$ contains edges wx with

- $w = x^1$ or
- $w = x^2$ or
- $w_1 = x_1 = 0$, $w = x^j$, $j \neq i$ and $j > 2$ or
- $w_1 = 1$, $w = (x^j)^{j-1}$, $j \neq i$ and $j - 1 \neq i$ and $j > 2$.

If $i = 2$ then $E(LTQ_n(s))$ contains edges wx with

- $w = x^1$ or
- $w_1 = x_1 = 0$, $w = x^j$ and $j > 2$ or
- $w_1 = 1$, $w = (x^j)^{j-1}$ and $j > 3$.

If we delete the bit on the i-th position of vertices in $E(LTQ_n(s))$, we get a subset of $E(LTQ_{n-1})$ described above.

The degree of every vertex $f \in V(LTQ_n(s))$ with $f_1 = 0$ is $n - 1$ because $LTQ_n(s)$ contains edges from f to all vertices f^j with $j \neq i$. The degree of every vertex $g \in LTQ_n(s)$ with $g_1 = 1$ is $n - 2$ because $LTQ_n(s)$ contains either edges to f^1, f^2 and $(f^j)^{j-1}$ with $2 < i \leq n$, $j \neq i$ and $j - 1 \neq i$ or edges to f^1 and $(f^j)^{j-1}$ with $n \geq j > 3$. The number of edges follows. □

2.7 Möbius cube

Definition. *The Möbius cube MQ_n of dimension n has the same vertex-set as Q_n and a vertex $x = x_n x_{n-1} \cdots x_1$ is connected to n other vertices y^i, $1 \leq i \leq n$, where each y^i satisfies:*

- $y^i = x_n x_{n-1} \ldots x_{i+1} \bar{x}_i x_{i-1} \ldots x_1$ *if* $x_{i+1} = 0$
- $y^i = x_n x_{n-1} \ldots x_{i+1} \bar{x}_i \bar{x}_{i-1} \ldots \bar{x}_1$ *if* $x_{i+1} = 1$

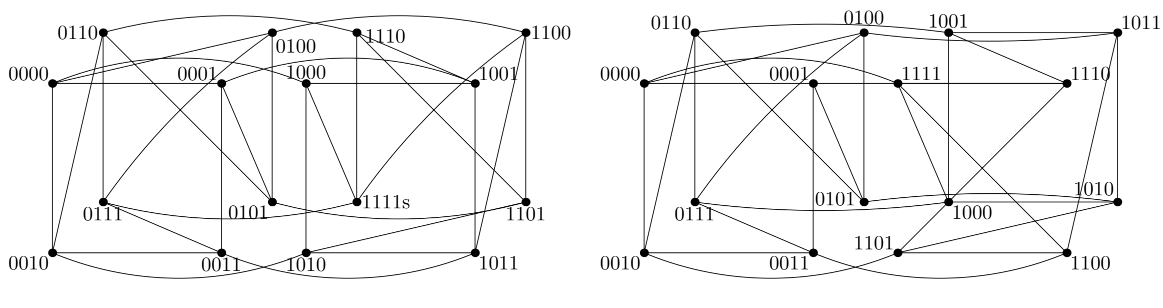

Figure 2.7: The Möbius cubes 0-MQ_4 and 1-MQ_4.

The presence of an edge between x and y^n and the value of x_{n+1} are undefined so we can assume x_{n+1} is either equal to 0 or to 1 which gives us two variants of MQ_n: 0-MQ_n for $x_{n+1} = 0$ and 1-MQ_n for $x_{n+1} = 1$.

MQ_n is edge-pancyclic for every $n \geq 2$ proved by Xu and Xu [46]. Xu et al. [44] showed that for every two vertices x and y with distance d there is an xy-path in MQ_n ($n \geq 3$) of every length from d to $2^n - 1$ except $d + 1$.

The subgraph of MQ_n (both variants) induced by vertices with the same prefix p of length k ending with 1 is isomorphic to 1-MQ_{n-k}. For a prefix ending with 0 it is 0-MQ_{n-k}.

Proposition 2.4. *The subgraph G of MQ_n induced by vertices with the same bit in a fixed coordinate i ($1 \leq i < n$) has 2^{n-1} vertices. The degree of a vertex $x = x_n x_{n-1} \cdots x_1$ in G is $n - k$ where $k = |\{j; x_j = 1, i+1 < j \leq n+1\}| + 1$.*

Proof. If we fix the coordinate i, all the edges of MQ_n that join the vertex $x \in V(G)$ with a vertex that differs in i-th coordinate are deleted. There is one such edge for each bit from the set $\{j; x_j = 1, i+1 < j \leq n+1\}$ because for every bit x_j with $j > i$ there is an edge between x and $y = x_n x_{n-1} \ldots x_j \bar{x}_{j-1} \bar{x}_{j-2} \ldots \bar{x}_1$. One more edge is deleted because there is an edge in MQ_n that changes either the bit in coordinate i for $x_{i+1} = 0$ or all bits with positions $1 \leq p \leq i$ for $x_{i+1} = 1$. $\square$

2.8 Balanced hypercube

The balanced hypercube of dimension n, denoted by BQ_n, was proposed by Wu and Huang [39]. Let $\oplus$ denote the modulo 4 addition and let $\ominus$ denote the modulo 4 subtraction.

Definition. *The BQ_n is a graph with the vertex-set $V = \{0, 1, 2, 3\}^n$. A vertex $a = a_n a_{n-1} \ldots a_1$ is connected to the following $2n$ vertices:*

- $a_n \ldots a_2(a_1 \oplus 1)$,
- $a_n \ldots a_2(a_1 \ominus 1)$,
- $a_n \ldots a_{i+1}(a_i \oplus (-1)^{a_n}) a_{i-1} \ldots (a_1 \oplus 1)$,
- $a_n \ldots a_{i+1}(a_i \oplus (-1)^{a_n}) a_{i-1} \ldots (a_1 \ominus 1)$

for every $1 < i \leq n$.

The balanced hypercube BQ_n is a bipartite graph. It is $2n$-regular and vertex transitive. Xu et al. [41] showed that BQ_n is edge-bipancyclic and hamiltonian laceable.

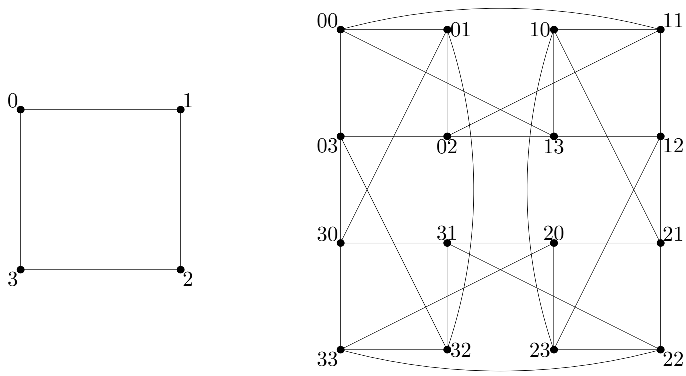

Figure 2.8: The balanced hypercubes BQ_1 and BQ_2.

3. Paths and cycles in faulty hypercubes

Applications of the hypercube in the theory of interconnection networks inspired questions about its robustness. If we remove the set of faulty (busy) vertices $F \subset V(Q_n)$ from Q_n, is there a path (several paths or a cycle) in $Q_n - F$ covering almost all remaining vertices? And how many vertices can be faulty in the worst case so that the desired property is preserved?

This problem is sometimes considered more generally with both faulty vertices and faulty edges. The number of faulty vertices will then be denoted by f_v and the number of faulty edges by f_e. The cardinality of F will be denoted by f.

3.1 Long paths

Let $F \subset V(Q_n)$ and $f = |F|$. A path in $Q_n - F$ is *long* if it contains at least $2^n - 2f - 2$ vertices. The problem of long paths in faulty hypercubes was first studied by Fu [19] who proved that there is a long path in $Q_n - F$ between any two vertices from $V(Q_n) - F$ if $f \leq n - 2$. Hung, Chang and Sun [26] proved that there exists a path of length at least $2^n - 2f$ in $Q_n - F$ if $f \leq n - 2$ and there is at least one faulty vertex in each bipartite class of Q_n.

The bound on the number of faulty vertices was improved by Hung et al. [27] to $f \leq 2n - 5$ if each vertex v of $Q_n - F$ has $deg(v) \geq 2$. Fink and Gregor [17] showed that significantly weaker conditions are sufficient for $f \leq 2n - 4$ (except for Q_4). They also introduced the first quadratic bound of $|F|$. They showed that for every set F of at most $(n^2 + n - 4)/4$ vertices in Q_n and $n \geq 5$, the graph $Q_n - F$ contains a long path between any two vertices such that each of them has at most 3 neighbors in F.

3.2 Long cycles

A cycle in $Q_n - F$ is *long* if its length is at least $2^n - 2f$. The existence of long cycles in faulty hypercube was probably first studied by Chan and Lee [2] who showed that there exists a cycle of length at least $2^n - 2f$ in Q_n if $f \leq \lfloor \frac{n+1}{2} \rfloor$. Moreover, they proposed a distributed algorithm that finds such a cycle. Tseng [36] studied cycles in hypercubes with both faulty edges and vertices. He proved that there is a long cycle in $Q_n - F$ if $f_v + f_e \leq n - 1$ and $f_e \leq n - 4$. These results were improved by Sengupta [35] who showed that there is a long cycle in $Q_n - F$ if $f_v + f_e \leq 2n - 4$ and either $f_e < n - 1$ or $f_v > 0$. This bound was improved by Fu [20] to $f_v \leq 2n - 4$ (and $f_e = 0$). The next improvement was made by Castañeda and Gotchev [4] who proved that $f_v \leq 3n - 7$ (and $f_e = 0$) and $n \geq 5$ is a sufficient bound.

All mentioned bounds on f_v are linear in n. First quadratic bound on f_v (and $f_e = 0$) was showed by Fink and Gregor in [17]. They showed that $Q_n - F$ contains a long cycle if $f \leq \frac{n^2}{10} + \frac{n}{2} + 1$ and $n \geq 15$. This result is asymptotically optimal. It was conjectured (Castañeda and Gotchev 2009) that for every set $F \subset V(Q_n)$

with $|F| \leq \binom{n}{2} - 2$ and $n \geq 4$ the graph $Q_n - F$ contains a cycle of length at least $2^n - 2f$ and that this bound is sharp. Fink and Gregor [18] proved this conjecture.

3.3 Routing

The problem of long paths may be generalized for several paths which connect two sets A and B of vertices of $Q_n - F$ where $A \neq B$ and $|A| = |B| = k$. AB-path is a path that connects some vertex of A with some vertex of B. AB-routing is a collection of k vertex-disjoint AB-paths. AB-routing $P_1, P_2, \ldots, P_k$ in $Q_n - F$ is *long* if

$$|P_1| + |P_2| + \cdots + |P_k| \leq 2^n - 2f - k - 1.$$

This definition corresponds to a long path for $k = 1$. Dvořák et al. [10] proposed that $Q_n - F$ contains a long AB-routing only if

$$b(A \cap B) + b(A \cup B) \leq 2$$

where $b(C) = ||C \cap X| - |C \cap Y||$ is the *balance* of C and X, Y are the bipartite classes of Q_n. A set is *monopartite* if it is a subset of one bipartite class.

Theorem 1 ([18]). *Let $n \geq 5$, $F \subset V(Q_n)$ and $A, B \subset V(Q_n - F)$ be such that $|F| \leq \lfloor \frac{n}{2} \rfloor$, $|A| = |B| = 2$ and $A \cup B$ is not monopartite, then $Q_n - F$ has a long AB-routing.*

3.4 Hamiltonicity, bipancyclicity and other properties

Several authors, for example Chen and Shin [6], showed the following result on the hypercube.

Theorem 3.1. *The graph $Q_n - F$ with $F \subset E(Q_n)$ and $n \geq 2$ contains a hamiltonian cycle if $|F| \leq n - 2$.*

Some other interesting results are listed below. Some of them will be useful in Chapter 5.

Theorem 3.2. (Li et al. [9]) *The graph $Q_n - F$ with $F \subset E(Q_n)$ and $n \geq 2$ contains a hamiltonian cycle provided that $|F| \leq 2n-5$ and each vertex is incident with at least two non-faulty edges.*

Theorem 3.3. (Xu et al. [45]) *For any two different vertices x and y with distance d in Q_n with $F \subset E(Q_n)$, if $|F| \leq n-2$ and $n \geq 2$, then Q_n contains a fault-free xy-path of length l for every l with $d+2 \leq l \leq 2n-1$, where l has the same parity as d.*

Theorem 3.4. (Li et al. [30]) *For every $n \geq 2$ and every $F \subset E(Q_n)$ with $|F| \leq n-2$ the graph $Q_n - F$ is bipancyclic.*

Theorem 3.5. (Chen [5]) *Let x and y be any two vertices in the n-dimensional hypercube Q_n with distance d and let $F \subset E(Q_n)$ with $|F| \leq 2n-5$ such that every vertex of $Q_n - F$ is incident with at least two edges. Then there exists an xy-path in $Q_n - F$ of each length $l \in \{l;\ s \leq l \leq 2n-1,\ l-s \text{ is even}\}$, where $s = h$ if $n-1 \leq h \leq n$, and $s = h+2$ if $n-4 \leq h \leq n-2$ and $h \geq 2$, and $s = h+4$ otherwise. Hence, the diameter of $Q_n - F$ is n.*

4. Hypercubes in augmented cubes

The augmented cube AQ_n contains many spanning subgraphs isomorphic to the hypercube Q_n. The aim of this chapter is to study how to find such subgraph G in a faulty AQ_n with $F \subseteq E(AQ_n)$ so that $|E(G) \cap F|$ is as small as possible.

We partition the edges of AQ_n into $2n-1$ sets, called *classes*. Let C_i with $2 \leq i \leq n$ denote the set of edges $uv \in E(AQ_n)$ such that $v = u_n \dots u_{i+1}\bar{u}_i u_{i-1} \dots u_1$. The class C_i consists of hypercube edges in dimension i. Let $C_{\leq i}$ with $1 \leq i \leq n$ denote the set of edges $uv \in E(AQ_n)$ such that $v = u_n \dots u_{i+1}\bar{u}_i \bar{u}_{i-1} \dots \bar{u}_1$. These classes consist of suffix edges. The class $C_{\leq 1}$ is treated as a class of suffix edges only for technical reasons. Clearly, "to be of the same class" is an equivalence relation on $E(AQ_n)$.

In this chapter we will need some elementary facts from linear algebra. We will refer to vertices of AQ_n as to vectors of the n-dimensional vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$. The elements of $\mathbb{F}_2^n$ are binary vectors of length n, and the space is endowed with modulo 2 component addition denoted by $\oplus$ and scalar multiplication by 0 and 1.

The *representing vector* of a class C_i (resp. $C_{\leq i}$), denoted by e_i (resp. $e_{\leq i}$), is the binary vector of length n such that $u = v \oplus e_i$ (resp. $u = v \oplus e_{\leq i}$) for every $uv \in C_i$ (resp. $uv \in C_{\leq i}$). Explicitly,

$$e_i = (v_n, v_{n-1}, \dots, v_1) \text{ where } v_j = 1 \text{ for } j = i \text{ and } v_j = 0 \text{ otherwise,}$$

and

$$e_{\leq i} = (w_n, w_{n-1}, \dots, w_1) \text{ where } w_j = 1 \text{ for } j \leq i \text{ and } w_j = 0 \text{ otherwise.}$$

We define $T_n = \{e_i; 2 \leq i \leq n\} \cup \{e_{\leq i}; 1 \leq i \leq n\}$.

A *class-preserving* subgraph S_n of AQ_n is a spanning subgraph of AQ_n such that $E(S_n)$ is a union of some classes of edges of AQ_n. The *representing set* $B(S_n)$ of S_n is the set $\{e; C_e \subset E(S_n)\}$ where C_e is a class of edges and e is its representing vector. The graph $S_n(B)$, where $B \subseteq T_n$, is the subgraph of AQ_n edge-induced by the set $\bigcup_{e \in B} C_e$ where C_e is the class with the representing vector e.

A *class basis* B is a basis of $\mathbb{F}_2^n$ such that $B \subseteq T_n$. The family of *class bases* of $\mathbb{F}_2^n$ is denoted by $\mathcal{B}_n$. The only basis of $\mathbb{F}_2^0$ is the empty bases so $\mathcal{B}_0 = \{\emptyset\}$.

In this chapter we will describe a construction of every class-preserving subgraph of AQ_n isomorphic to Q_n. We discuss the existence of non-class-preserving subgraphs of AQ_n isomorphic to Q_n. We will need the following characterization of the hypercube proposed by Saad and Schulz [34].

Lemma 4.1. ([34]) *A graph $G = (V, E)$ is isomorphic to Q_n if and only if*

1. $|V| = 2^n$,
2. $\deg(v) = n$ *for every* $v \in V$,
3. G *is connected,*
4. *Every subgraph induced by all neighbors of two adjacent vertices u and v has a perfect matching whose every edge joins a neighbor of u with a neighbor of v.*

Now we characterize class-preserving subgraphs of AQ_n isomorphic to Q_n.

Lemma 4.2. *Let S_n be a class-preserving subgraph of AQ_n. Then S_n is isomorphic to Q_n if and only if $B(S_n) \in \mathcal{B}_n$.*

Proof. Assume that S_n is isomorphic to Q_n. The number of edges of each class in AQ_n is 2^{n-1}. Since $|E(Q_n)| = n2^{n-1}$, it follows that the number of used classes must be n; that is $|B(S_n)| = n$. Suppose that $B(S_n)$ is not a generating set of $\mathbb{F}_2^n$. Then there is a vector $v \in \mathbb{F}_2^n$ that cannot be obtained by a linear combination of the vectors of $B(S_n)$. It follows that there is no path between 0^n and v in S_n, a contradiction. Thus, $B(S_n)$ is the basis of $\mathbb{F}_2^n$.

Let $B \in \mathcal{B}_n$. To prove that the graph $S_n(B)$ is isomorphic to Q_n, we apply Lemma 4.1. It suffices to check the following conditions:

1. The vector space $\mathbb{F}_2^n$ has cardinality $|\mathbb{F}_2^n| = 2^n$.

2. Every vertex is incident with exactly one edge of each class with its representing vector in B and $|B| = n$.

3. Every vector v of $\mathbb{F}_2^n$ can be obtained by a sum of vectors $A \subseteq B$. This means that there is a path of length $|A|$ in $S_n(B)$ from 0^n to v using exactly one edge of every class C_f with $f \in A$.

4. Let a, b be two adjacent vertices of $S_n(B)$ and $a \oplus b = l$ and let $c = a \oplus m$ where $m \in B$, $m \neq l$ is a representing vector of some class. Then there is a vertex $d \in V(G)$ with $d = b \oplus m$. Then $c \oplus m \oplus l \oplus m = d$ so $c \oplus l = d$ which means that c and d are connected with an edge of class C_l. This induces the desired perfect matching.

□

In Lemmas 4.3 and 4.4 we will describe a construction of all class bases of $\mathbb{F}_2^n$. We define three operations that derive a class basis of $\mathbb{F}_2^n$ from a class basis of $\mathbb{F}_2^{n-1}$. Recall that $0b$ where $b \in \mathbb{F}_2^{n-1}$ is the vector $c \in \mathbb{F}_2^n$ with $c_n = 0$ and $c_i = b_i$ for every $1 \leq i \leq n$.

Definition 1. *For $B \in \mathcal{B}_{n-1}$ where $n \geq 1$ let*

- $\sigma_n B = \{0b; b \in B\} \cup \{e_n\}$ *if* $n \geq 2$,
- $\sigma_{\leq n} B = \{0b; b \in B\} \cup \{e_{\leq n}\}$ *if* $n \geq 1$,
- $\sigma_{\leq n,n} B = \{0b; b \in B, b \neq 1^{n-1}\} \cup \{e_n, e_{\leq n}\}$ *if* $1^{n-1} \in B$ *and* $n \geq 2$.

For example, $\sigma_{\leq 1}\emptyset = \{1\}$, $\sigma_{\leq 2}\{1\} = \{01, 11\}$, $\sigma_2\{1\} = \{01, 10\}$ and $\sigma_{\leq 2,2}\{1\} = \{10, 11\}$.

Lemma 4.3. *If B is a class basis of $\mathbb{F}_2^{n-1}$, then $\sigma_n B$, $\sigma_{\leq n} B$ and $\sigma_{\leq n,n} B$ (if defined) are class bases of $\mathbb{F}_2^n$.*

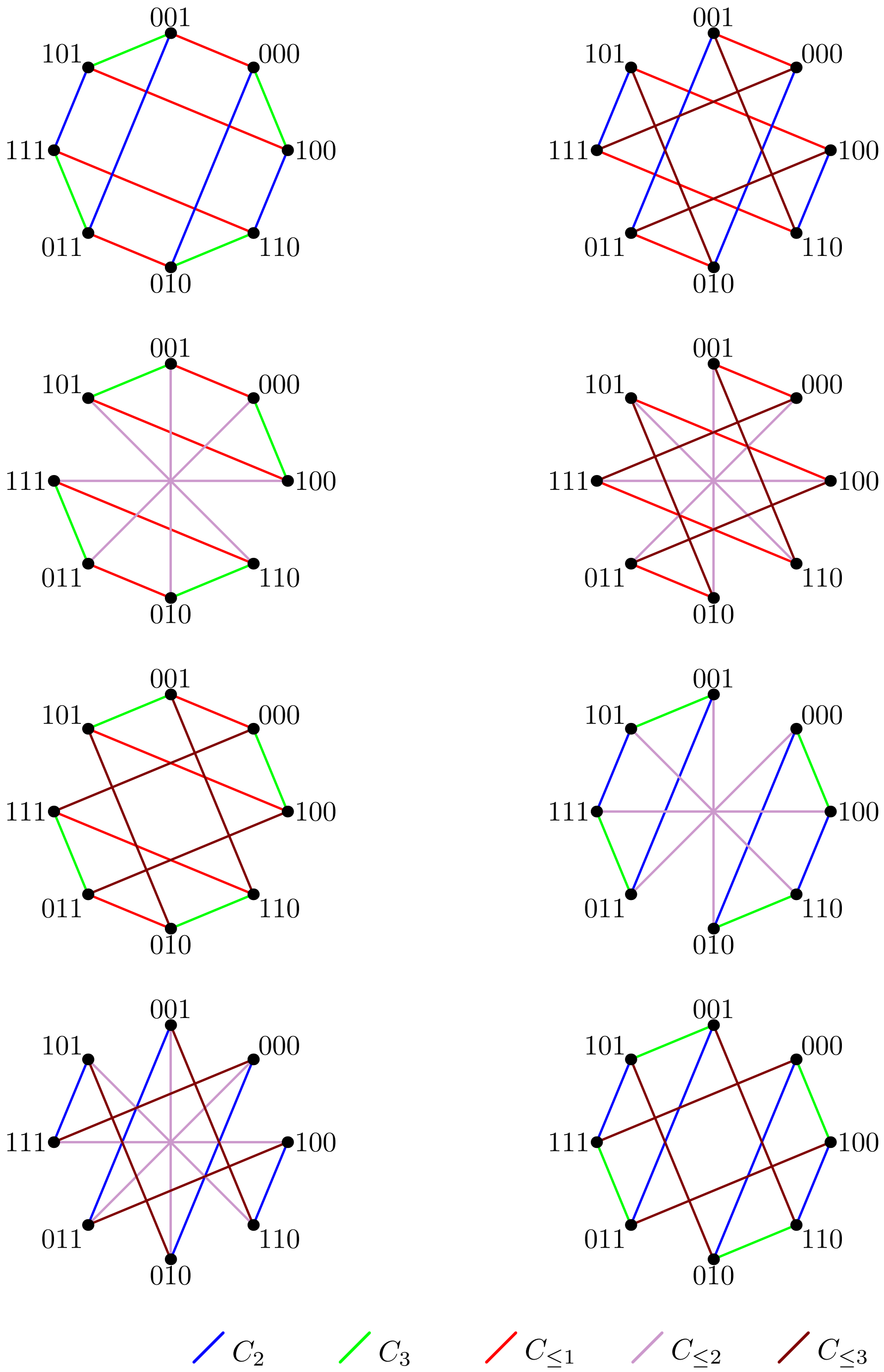

Figure 4.1: All class-preserving subgraphs of AQ_3 isomorphic to Q_3 and their representing sets.

Proof. Each of the sets $\sigma_n B$, $\sigma_{\leq n} B$, $\sigma_{\leq n,n} B$ is clearly linearly independent. We show that they are also generating sets of $\mathbb{F}_2^n$.

In the first case, every vector $0v \in \mathbb{F}_2^n$ is a linear combination of vectors $\{0b; b \in B\}$ and $1v = 0v \oplus e_n$.

In the second case, every vector $0v \in \mathbb{F}_2^n$ is a linear combination of vectors $\{0b; b \in B\}$ and $1v = 0\bar{v} \oplus e_{\leq n}$.

In the last case, for every vector bv where $b \in \mathbb{F}_2$ and $v \in \mathbb{F}_2^{n-1}$ either $0v$ or $1v$ is a linear combination of vectors $\sigma_{\leq n,n} B \setminus \{e_n, e_{\leq n}\}$ and $bv = 0v = 1v \oplus e_n$ or $bv = 1v = 0v \oplus e_n$, respectively. $\square$

Lemma 4.4. *Every $B \in \mathcal{B}_n$ for $n \leq 1$ is constructed from some $A \in \mathcal{B}_{n-1}$ using one of the operations from Definition 1. The only basis in $\mathcal{B}_0$ is the empty set.*

Proof. Clearly the only basis of the space $\mathbb{F}_2^0$ is the empty basis. To prove that every basis $B \in \mathcal{B}_n$ with $n \geq 1$ can be constructed by the described operations, we find a unique basis $A \in \mathcal{B}_{n-1}$ and an operation to derive B. Let B be a class basis of $\mathbb{F}_2^n$. It may either contain one or two vectors v with $v_n = 1$ (otherwise it is not a class basis). If it contains one such vector (that is, e_n or $e_{\leq n}$), we can delete it and remove leading zero from the other vectors and we receive a basis $A \in \mathcal{B}_{n-1}$. Hence, $B = \sigma_n A$ or $B = \sigma_{\leq n} A$. If it contains two such vectors (both e_n and $e_{\leq n}$), we can replace them with $01^{n-1} = 0e_{\leq n-1} = (e_n \oplus e_{\leq n})$ (which is clearly linearly independent with the other vectors) and then we remove the leading zero from all the vectors and we receive a basis $A \in \mathcal{B}_{n-1}$. Then $B = \sigma_{\leq n,n} A$.

We showed that every class basis of $\mathbb{F}_2^n$ can be constructed from a unique class basis of $\mathbb{F}_2^{n-1}$ using the above operations. $\square$

Corollary 4.5. $|\mathcal{B}_0| = |\mathcal{B}_1| = 1$, $|\mathcal{B}_{n+2}| = 3|\mathcal{B}_{n+1}| - |\mathcal{B}_n|$ *for every* $n \geq 0$.

Proof. The only basis of the space $\mathbb{F}_2^0$ is the empty basis so $|\mathcal{B}_0| = 1$. The space $\mathbb{F}_2^1$ has the only class basis $\{e_{\leq 1}\}$ so $|\mathcal{B}_1| = 1$.

In $\mathcal{B}_{n+1}$ there are exactly $|\mathcal{B}_n|$ bases that do not contain the vector $e_{\leq n+1}$, these bases were derived from $\mathcal{B}_n$ using the σ_{n+1} operation.

Two new class bases of $\mathbb{F}_2^{n+2}$ are formed from every class basis of $\mathcal{B}_{n+1}$ that does not contain $e_{\leq n+1}$. Three new class bases are formed from every class basis of $\mathcal{B}_{n+1}$ that contains $e_{\leq n+1}$. All these bases are distinct. Therefore,

$$|\mathcal{B}_{n+2}| = 2|\mathcal{B}_n| + 3(|\mathcal{B}_{n+1}| - |\mathcal{B}_n|).$$

$\square$

Corollary 4.6. *The explicit number of class-preserving subgraphs of AQ_n isomorphic to Q_n is*

$$|\mathcal{B}_n| = -\frac{5 + 3\sqrt{5}}{5(3 + \sqrt{5})} \left(\frac{1}{2}(3 - \sqrt{5})\right)^n + \frac{1}{\sqrt{5}} \left(\frac{1}{2}(3 + \sqrt{5})\right)^n.$$

Proof. We use the method described by Epp [14]. The characteristic equation of $|\mathcal{B}_{n+2}| = 3|\mathcal{B}_{n+1}| - |\mathcal{B}_n|$ is $t^2 - 3t + 1 = 0$. This equation has two distinct roots $\frac{1}{2}(3 - \sqrt{5})$ and $\frac{1}{2}(3 + \sqrt{5})$. Now we have to find c and d such that $|\mathcal{B}_n| =$

$c\left(\frac{1}{2}(3-\sqrt{5})\right)^n + d\left(\frac{1}{2}(3+\sqrt{5})\right)^n$. To find such c and d we use the first two members of the sequence $(|\mathcal{B}_i|)_i$.

$$
\begin{aligned}
|\mathcal{B}_0| &= c\left(\frac{1}{2}(3-\sqrt{5})\right)^0 + d\left(\frac{1}{2}(3+\sqrt{5})\right)^0 \\
|\mathcal{B}_1| &= c\left(\frac{1}{2}(3-\sqrt{5})\right)^1 + d\left(\frac{1}{2}(3+\sqrt{5})\right)^1
\end{aligned}
$$

The solution of this system of equations is $c = -\frac{5+3\sqrt{5}}{5(3+\sqrt{5})}$ and $d = \frac{1}{\sqrt{5}}$. □

Lemma 4.7. *For every $v \in T_n$ there are two class bases $B_1, B_2 \in \mathcal{B}_n$ such that $B_1 \cap B_2 = \{v\}$.*

Proof. We will construct B_1, B_2 using operations from Definition 1. We put $B_1 = \{e_j; 2 \le j \le n\} \cup \{e_{\le i}\}$, which is obtained from the empty basis of $\mathbb{F}_2^0$ by one application of $\sigma_{\le 1}$, $i-1$ applications of $\sigma_{\le k,k}$ and $n-i$ applications of σ_l where $k = 2, \ldots, i$ and $l = i+1, \ldots, n$.

- If $v = e_{\le i}$ for some $1 \le i \le n$, then we put $B_2 = \{e_{\le j}; 1 \le j \le n\}$, which is obtained from the empty basis by n applications of $\sigma_{\le k}$ where $k = 1, \ldots, n$.
- Otherwise; $v = e_i$ for some $2 \le i \le n$, and we put $B_2 = \{e_{\le j}; 1 \le j \le n, j \ne i\} \cup \{e_i\}$, which is obtained from the empty basis by $i-1$ applications of $\sigma_{\le k}$, one application of σ_i, and another $n-i$ applications of $\sigma_{\le l}$ where $k = 1, \ldots, i-1$ and $l = i+1, \ldots, n$.

Note that $B_1 \cap B_2 = \{v\}$ in both cases. Since B_1, B_2 are obtained from the empty basis by operations from Definition 1, we have $B_1, B_2 \in \mathcal{B}_n$ by Lemma 4.3. □

To generalize Lemma 4.9 we will need well-known Steinitz exchange theorem.

Theorem 4.8. (Steinitz exchange theorem) *Let V be a vector space. Let $G = \{v_1, \ldots, v_n\}$ be a generating set of V and let $X = \{x_1, \ldots, x_k\}$ be a set of linearly independent vectors from V. Then $k \le n$ and there exists a permutation π of $\{1, \ldots, n\}$ such that $\{x_1, \ldots, x_k, v_{\pi(k+1)}, \ldots, v_{\pi(n)}\}$ is a generating set of V.*

Lemma 4.9. *Every linearly independent set $A \subset T$ with $1 \le |A| < n$ can be expanded to two different bases $B_1, B_2 \in \mathcal{B}_n$ such that $B_1 \cap B_2 = A$.*

Proof. Let $e \in A$ and let $\bar{B}_1 = \{\bar{a}_1, \ldots \bar{a}_n\}, \bar{B}_2 = \{\bar{b}_1, \ldots \bar{b}_n\}$ be two class bases such that $\bar{B}_1 \cap \bar{B}_2 = \{e\}$ obtained by Lemma 4.7. Then by Theorem 4.8 we have generating sets $B_1 = A \cup \{\bar{a}_{\pi_1(k+1)}, \ldots, \bar{a}_{\pi_1(n)}\}$ and $B_2 = A \cup \{\bar{b}_{\pi_2(k+1)}, \ldots, \bar{b}_{\pi_2(n)}\}$ with $|B_1| = |B_2| = n$, so they are bases. The bases B_1 and B_2 contain only vectors e_i and $e_{\le i}$ so $B_1, B_2 \in \mathcal{B}_n$. Clearly $B_1 \cap B_2 = A$. □

Theorem 4.10. *For every $F \subseteq E(AQ_n)$, $n \ge 1$, and linearly independent A, $\emptyset \ne A \subseteq T_n$, there is B, such that $A \subseteq B \in \mathcal{B}_n$ and $|E(S_n(B)) \cap F| \le \frac{f+f'}{2}$ where $f = |F|$ and $f' = |E(S_n(A)) \cap F|$.*

Proof. If $A \in \mathcal{B}_n$ then we have $B = A$ and $|E(S_n(B)) \cap F| = f' \leq \frac{f+f'}{2}$. Otherwise, Lemma 4.9 implies that we can find two class bases B_1 and B_2 of $\mathbb{F}_2^n$ such that $B_1 \cap B_2 = A$. Without loss of generality we assume that $|E(S_n(B_1)) \cap F| \leq |E(S_n(B_2)) \cap F|$. Then the subgraph $S_n(B_1)$ contains at most

$$\frac{f-f'}{2} + f' = \frac{f+f'}{2}$$

faulty edges. □

Theorem 4.11. *For every $F \subseteq E(AQ_n)$ there is a class-preserving subgraph G in AQ_n isomorphic to Q_n such that $|E(G) \cap F| \leq \frac{n}{2n-1}|F|$.*

Proof. Let C be a class of edges of AQ_n with lowest $|C \cap F|$. It is clear that $|C \cap F| \leq \frac{|F|}{2n-1}$. We put $A = \{e\}$ where e is the representing vector of C. Then by Theorem 4.10 we find B such that $A \subseteq B \in \mathcal{B}_n$ and $|E(S_n(B)) \cap F| = \frac{|F| + \frac{|F|}{2n-1}}{2} = \frac{n}{2n-1}|F|$. □

Theorem 4.12. *For every $n \geq 4$ and $F \subseteq E(AQ_n)$ with $|F| \leq 3n-7$ such that $\delta(AQ_n - F) \geq 2$ the augmented cube AQ_n contains a class-preserving subgraph G isomorphic to Q_n such that $\delta(G - F) \geq 2$.*

Proof. The degree of vertices in Q_n is smaller by $n-1$ than the degree of vertices in AQ_n. Thus, only the vertices incident with at most n non-faulty edges in AQ_n can be incident with less than two non-faulty edges in G. If at least vertices in $AQ_n - F$ had the degree at most n then we would have $|F| \geq 3n-6$. Thus, at most two vertices of AQ_n are incident with less than $n+1$ non-faulty edges. The three following situations may occur:

1. There is no vertex $v \in V(AQ_n)$ incident with less than $n+1$ non-faulty edges.

   Every vertex of every spanning subgraph isomorphic to Q_n is incident with at least two non-faulty edges. By Theorem 4.11 we find G such that for

   $$|E(G) \cap F| \leq \frac{n}{2n-1}(3n-7)$$

   and for $n \geq 4$
   $$\frac{3n^2-7n}{2n-1} \leq 2n-5.$$

2. There is exactly one vertex $v \in V(AQ_n)$ with $2 \leq d = deg_{AQ_n - F}(v) \leq n$.

   The vertex v is incident with $2n-1-d$ faulty edges. There are at most $n-6+d$ faulty edges in $AQ_n - \{v\}$. Let $D = \{e \in T_n; \{v, v \oplus e\} \notin F\}$ and let $a, b \in D$ be the representing vectors of distinct classes C_a, C_b with $|C_a \cap F| \leq \frac{n-6+d}{d}$ and $|C_b \cap F| \leq \frac{n-6+d}{d}$. Then we put $A = \{a, b\}$ and by Theorem 4.10, we have B such that $A \subseteq B \in \mathcal{B}_n$ and

   $$|E(S_n(B)) \cap F| \leq \frac{3n-7+2\frac{n-6+d}{d}}{2} = \frac{3n-5+2\frac{n-6}{d}}{2} \leq \frac{4n-11}{2} \leq 2n-5.$$

   The second inequality holds since $d \geq 2$. The vertex v is incident with at least two non-faulty edges $\{v, v \oplus a\}$, $\{v, v \oplus b\}$ in G.

3. There are exactly two vertices $u, v \in V(AQ_n)$ incident with less than $n+1$ non-faulty edges.

   Let $D_1 = \{e \in T_n; \{u, u \oplus e\} \notin F\}$, $D_2 = \{e \in T_n; \{v, v \oplus e\} \notin F\}$, $d_1 = |D_1|$ and $d_2 = |D_2|$. Without loss of generality we assume that $d_1 \leq d_2$. One of the following two situations may occur:

   (a) $D_1 \cap D_2 \neq \emptyset$. Let $t \in D_1 \cap D_2$.
   There are at least $4n-3$ edges incident with u or v. Since $n \geq 4$ and $d_1 + d_2 \geq 4n - 3 - (3n-7) = n+4$, we have $d_2 \geq 4$. For every $t_1 \in D_1$, $t_1 \neq t$ there is $t_2 \in D_2$, $t_2 \neq t$ such that either

   - $t_1 = t_2$ or
   - $t_1 \neq t_2$ and the set $\{t, t_1, t_2\}$ is linearly independent.

   The vertices u, v may be connected with a faulty edge and the edges $v(v \oplus t_1)$ and $u(u \oplus t_2)$ may also be faulty, so we have

$$\begin{aligned} |E(S_n(A)) \cap F| &= f' \leq |F| - (2n - 1 - d_1 + 2n - 1 - d_2 - 1) + 2 = \\ &= |F| - 4n - 1 - (d_1 + d_2) \leq n - 2. \end{aligned}$$

   So by Theorem 4.10 we find $B \in \mathcal{B}_n$ such that $A \subseteq B$ and

$$|E(S_n(B)) \cap F| \leq \frac{3n - 7 + n - 2}{2} = 2n - 4.5$$

   and because $|E(S_n(B)) \cap F|$ is an integer, it is clear that $|E(S_n(B)) \cap F| \leq 2n - 5$.

   (b) $D_1 \cap D_2 = \emptyset$.
   If $d_1 = d_2 = n$ then the intersection $D_1 \cap D_2$ would not be empty so $d_1 + d_2 \leq 2n-1$. Because $n \geq 4$ and $d_1 + d_2 \geq 4n - 3 - (3n-7) = n+4$, we have $d_2 \geq 4$. Note that the only two combinations of three linearly dependent vectors in T_n are $\{e_{\leq i}, e_{i+1}, e_{\leq i+1}\}$ and $\{e_{\leq i}, e_i, e_{\leq i+1}\}$. The only combination of four linearly dependent vectors from T_n is $\{e_{\leq i}, e_{i+1}, e_{i+2}, e_{\leq i+2}\}$. In the worst case, the set of two vectors from T_n can be expanded to two distinct linearly dependent sets of at most four vertices. Then for every $t_1, t_2 \in D_1$ we find $t_3, t_4 \in D_2$ such that $t_1 \neq t_2$ and $t_3 \neq t_4$ and $A = \{t_1, t_2, t_3, t_4\}$ is linearly independent. The edges $u(u \oplus t_3)$,$u(u \oplus t_4)$,$v(v \oplus t_1)$,$v(v \oplus t_2)$ may be faulty. (This is why there is $+4$ in the next inequality.) Then, if $uv \notin E(AQ_n)$, we have

$$\begin{aligned} |E(S_n(A)) \cap F| &= f' \leq |F| - (2n - 1 - d_1 + 2n - 1 - d_2) + 4 \leq \\ &\leq |F| - (2n - 1) + 4 \leq n - 2. \end{aligned}$$

   If $uv \in E(AQ_n)$ then $uv \in F$ since $D_1 \cap D_2 = \emptyset$ and $D_1, D_2 \subseteq T_n - \{u \oplus v\}$. If $d_1 = n-1$ and $d_2 = n$, the intersection $D_1 \cap D_2$ would not be empty. So we have $d_1 + d_2 \leq 2n - 2$ and

$$\begin{aligned} |E(S_n(A)) \cap F| &= f' \leq |F| - (2n - 1 - d_1 + 2n - 1 - d_2 - 1) + 4 \leq \\ &\leq |F| - (2n - 1) + 4 \leq n - 2. \end{aligned}$$

By Theorem 4.10 we find $B \in \mathcal{B}_n$ such that $A \subseteq B$ and

$$|E(S_n(B)) \cap F| \leq \frac{3n-7+n-2}{2} = 2n - 4.5$$

and because $|E(S_n(B)) \cap F|$ is an integer, it is clear that $|E(S_n(B)) \cap F| \leq 2n - 5$.

□

All subgraphs of AQ_n isomorphic to Q_n are not class-preserving subgraphs of AQ_n. Clearly AQ_2 does not contain any non-class preserving subgraph isomorphic to Q_2. Non-class-preserving subgraphs of AQ_3 isomorphic to Q_3 are depicted in Figure 4.2. We believe that these are all such subgraphs. By composition of these mappings of Q_3 to AQ_3 and the mappings obtained by Lemmas 4.3 and 4.4 we are able to obtain many more subgraphs of AQ_n isomorphic to Q_n. The method of composition of two such mappings is showed in Chapter 6.

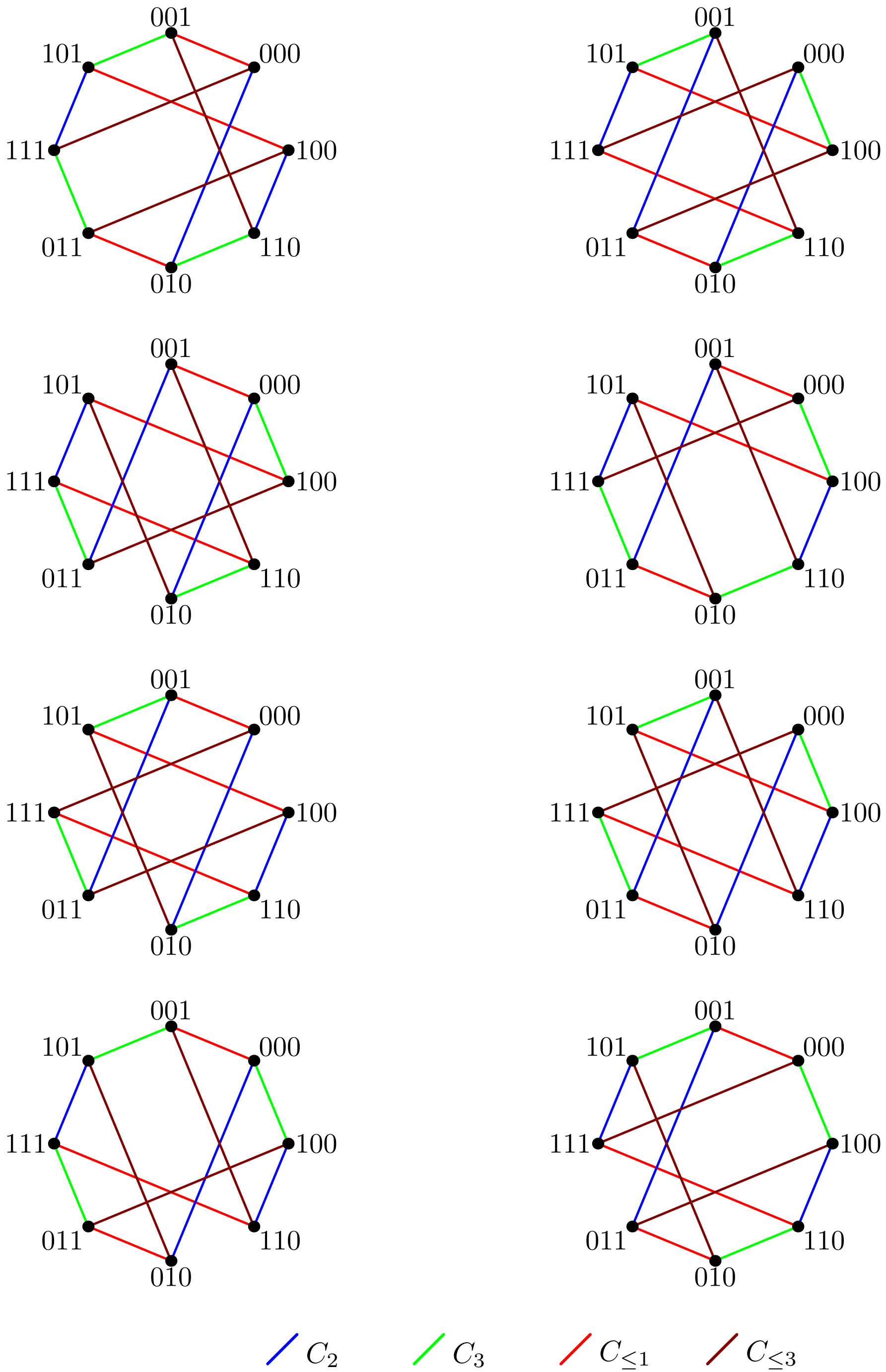

Figure 4.2: Non-class-preserving subgraphs of AQ_3 isomorphic to Q_3.

5. Paths and cycles in faulty augmented cubes

5.1 Previous results

Path and cycles in nonfaulty augmented cubes were first studied by Choudum and Sunitha [7]. They showed that AQ_n is pancyclic for every $n \geq 2$. They did not study the case of faulty vertices or edges.

Paths and cycles in faulty AQ_n were first studied by Hsu et al. [24]. They proved that $AQ_n - F$ is hamiltonian for every $F \subset V(AQ_n) \cup E(AQ_n)$ and every $n \geq 4$ with $|F| \leq 2n-3$. Similar result may also be proved directly from Theorem 4.11 and Theorem 3.1. They also showed that AQ_n is hamiltonian connected if $|F| \leq 2n-4$. And they proved that these bounds are tight.

Hsieh and Shiu [23] improved previous results of Choudum and Sunitha by proving that AQ_n is vertex-pancyclic for every $n \geq 2$. These results were improved by Ma et al. [32]. They showed that $AQ_n - F$ is pancyclic for every $F \subset E(AQ_n)$ with $|F| \leq 2n-3$. Wang et al. [37] showed a similar result for AQ_n with both faulty vertices and edges. Specifically, they showed that $AQ_n - F$ is pancyclic for every $F \subset E(AQ_n) \cup V(AQ_n)$ with $|F| \leq 2n-3$.

Fu [21] proved that $AQ_n - F$ is vertex-pancyclic for every $F \subset E(AQ_n)$ and $n \geq 2$ with $|F| \leq n-1$. Liu et al. [31] showed that every edge of $AQ_n - F$ lies on a hamiltonian cycle of $AQ_n - F$ where $F \subset E(AQ_n)$ if $|F| \leq 2n-3$.

5.2 New results

All results in this section are direct corollaries of Theorem 4.11 and previously known theorems from Chapter 3.

Theorem 5.1. *The graph $AQ_n - F$ is hamiltonian for every $F \subset E(AQ_n)$ and every $n \geq 4$ with $|F| \leq 2n-3$.*

Proof. By Theorem 4.11 there is a spanning subgraph G isomorphic to Q_n in AQ_n with $|E(G) \cap F| \leq n-2$. The graph $G - F$ is hamiltonian according to Theorem 3.1. Hamiltonian cycle in $G - F$ is clearly a hamiltonian cycle in $AQ_n - F$. Thus, $AQ_n - F$ is hamiltonian. □

Theorem 5.2. *The graph $AQ_n - F$ is hamiltonian for every $F \subset E(AQ_n)$ and every $n \geq 4$ with $|F| \leq 3n-7$ and $\delta(AQ_n - F) \geq 2$.*

Proof. By Theorem 4.12 there is a spanning subgraph G isomorphic to Q_n in AQ_n with $|E(G) \cap F| \leq 2n-5$ and $\delta(G - F) \geq 2$. The graph $G - F$ is hamiltonian according to Theorem 3.2. A hamiltonian cycle in $G - F$ is clearly a hamiltonian cycle in $AQ_n - F$. Thus, $AQ_n - F$ is hamiltonian. □

Theorem 5.3. *The graph $AQ_n - F$ with $F \subset E(AQ_n)$ and $n \geq 2$ contains a hamiltonian cycle provided that $|F| \leq 4n-11$ and each vertex is incident with at least $n+1$ non-faulty edges.*

Proof. Using Theorem 4.11 we find a subgraph G of AQ_n isomorphic to the hypercube Q_n with $|E(G) \cap F| \leq 2n-5$. The degree of each vertex in $AQ_n - F$ was at least $n+1$ and we removed $n-1$ edges from each vertex so the minimal degree of vertices in $G - F$ is 2. Then the result follows directly from Theorem 3.2. □

6. Composing monomorphisms to augmented cubes

We can find more subgraphs of AQ_n isomorphic to hypercube applying the following result. Moreover, it is possible to construct monomorphisms of many other graphs to AQ_n. If the binary string s is expressed by a formula f more complicated than one letter, we use the notation $(f)_i = s_i$.

Definition 2. *For mappings $g_1 : V_1 \to \{0,1\}^m$ and $g_2 : V_2 \to \{0,1\}^n$, $n, m \geq 1$ we define a composed map $h = h_{g_1,g_2} : V_1 \times V_2 \to \{0,1\}^{m+n}$ as follows. For $(w, x) \in V_1 \times V_2$ let*

- *$h(w, x) = g_1(w)g_2(x)$ if $(g_1(w))_1 = 0$,*
- *$h(w, x) = g_1(w)\overline{g_2(x)}$ if $(g_1(w))_1 = 1$.*

Lemma 6.1. *If g_1 is a monomorphism of G_1 to AQ_m and g_2 is a monomorphism of G_2 to AQ_n then the composed map h_{g_1,g_2} is a monomorphism of $G_1 \,\Box\, G_2$ to AQ_{m+n}.*

Proof. We have to prove that $h = h_{g_1,g_2}$ is injective and that it preserves edges.

- Assume that there is a vertex $uv \in AQ_{m+n}$, $u \in \{0,1\}^m$, $v \in \{0,1\}^n$ such that there are two vertices $(w, x), (y, z) \in V(G_1 \,\Box\, G_2)$, $(w, x) \neq (y, z)$ with $h(w, x) = h(y, z) = uv$. One of the following two situations may happen.
  - $w \neq y$. Then $g_1(w) \neq g_1(y)$ since g_1 is injective, so $h(w, x) \neq h(y, z)$, a contradiction.
  - $w = y$ and $x \neq z$. Then $g_2(x) \neq g_z(z)$ since g_2 is injective, so $h(w, x) \neq h(y, z)$, a contradiction.

  Thus, h is injective.

- Let $\{(w, x), (y, z)\} \in E(G_1 \Box G_2)$. By the definition of the Cartesian product, one of the following situations may occur:
  - $w = y$ and $x \neq z$.
    Then $g_1(w) = g_1(y)$ and $\{g_2(x), g_2(z)\} \in E(AQ_n)$ which means that vertices $h(w, x)$, $h(y, z)$ differ exactly in one of last n bits or exactly in a suffix of length at most n, so $\{h(w, x), h(y, z)\} \in E(AQ_{m+n})$.
  - $w \neq y$ and $x = z$.
    The edge $\{g_1(w), g_2(y)\} \in AQ_m$ is either a hypercube edge or a suffix edge. Recall that an edge of the class $C_{\leq 1}$ is a suffix edge. If $\{g_1(w), g_2(y)\}$ is a hypercube edge of AQ_m then $(g_1(w))_1 = (g_2(y))_1$ and $\{h(w, x), h(y, z)\}$ is a hypercube edge of AQ_{m+n}. If $\{g_1(w), g_2(y)\}$ is a suffix edge of AQ_m then $(g_1(w))_1 \neq (g_2(y))_1$ and $\{h(w, x), h(y, z)\}$ is a suffix edge of AQ_{m+n}.

  Thus, for every edge $uv \in E(G_1 \,\Box\, G_2)$ it holds $h(u)h(v) \in E(AQ_{m+n})$.

$\square$

Conclusion

The main contribution of this thesis are new results on hamiltonicity of augmented cubes with respect to faulty edges stated in Theorems 5.3, 5.1 and 5.2.

The bound on $|F|$ in Theorem 4.11 appears to be tight because it transfers previous tight results on hypercubes to previous tight results on augmented cube. Maybe better results could be obtained for more specific F.

Although many subgraphs of AQ_n isomorphic to Q_n are class-preserving subgraphs and they can be constructed applying the described method, these are not all the subgraphs of AQ_n isomorphic to Q_n. More such subgraphs can be obtained by composition of monomorphisms of class-preserving subgraphs and non-class preserving subgraphs. Composition of monomorphisms to augmented cubes has been described in Theorem 6.1. We suppose that these hypercubes are still not all subgraphs of AQ_n isomorphic to Q_n. As a further direction, one may ask for all subgraphs of AQ_n isomorphic to Q_n.

The bound on $|F|$ in Theorem 5.3 is probably not tight. Clearly, the tight bound is lower than $4n-9$ (see Figure 6.1). According to Theorem 3.2, the hypercube $Q_n - F$ is hamiltonian for every F with $|F| \leq 2n-5$ such that every vertex of Q_n is incident with at least two edges. By Theorem 4.11 AQ_n with $F \subset E$ and $|F| \leq 4n-11$ contains a subgraph G isomorphic to Q_n with $|G \cap F| \leq 2n-5$. We suppose that there is a way to preserve a minimal degree in $Q_n - F$ with similar bound on $|F|$.

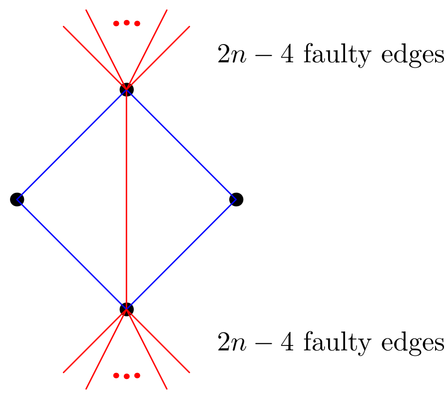

Figure 6.1: A counterexample to hamiltonicity of AQ_n with $4n-9$ faulty edges.

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